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# 1D action and partition function for the 2D Ising model with a boundary magnetic field

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**Abstract.** In this article we present an alternative method to that developed by B. McCoy and T.T. Wu to obtain some exact results for the 2D Ising model with a general boundary magnetic field and for a finite size system. This method is a generalisation of ideas from V.N. Plechko presented for the 2D Ising model in zero field, based on the representation of the Ising model using a Grassmann algebra. A Gaussian 1D action is obtained for a general configuration of the boundary magnetic field. When the magnetic field is homogeneous, we check that our results are in agreement with McCoy and Wu's previous work. This 1D action is used to compute in an efficient way the free energy in the special case of an inhomogeneous boundary magnetic field. This method is useful to obtain new exact results for interesting boundary problems, such as wetting transitions.

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## 1. Introduction

Studied for the first time in 1925 [1], the Ising model is one of the most important models of statistical physics. The two dimensional case, solved exactly for the first time in 1944 by Onsager [2], is the prototype of systems with second order phase transition and non Gaussian critical exponents. It has therefore been studied extensively by various exact and approximate methods. One important result is when Schultz, Mattis and Lieb [3, 4, 5] used transfer matrix method, Jordan-Wigner transformation and fermionization to simplify the Onsager solution in zero magnetic field, whereas fruitful links with 1D Quantum Field Theory and Conformal Field Theory have been developed in a more recent period to provide a more general frame for studying 2D critical systems. In particular, a way to express the Ising Hamiltonian as a Gaussian Grassmannian action was established long time ago [6, 7], and this idea was extended by Plechko to compute the partition function of the 2D Ising model in zero field for a large class of lattices [8, 9], using operator symmetries that simplify the algebra of transfer matrix.

The 2D Ising model has also been a starting point to study some boundary problems, for example the effect of a boundary magnetic field (BMF) on the propagation of a domain wall which is similar to a wetting or pinning problems. It is quite remarkable that exact results can be obtained in this case since the model with a general uniform field has not yet been solved except at the critical temperature [10, 11] where conserved quantities have been found. In a series of papers, McCoy and Wu computed the partition function associated with a uniform BMF on one side of a square lattice and evaluated the boundary correlation functions, using dimer statistics [12, 13, 14, 15, 16, 17]. With transfer matrix and spinor methods, some exact results have also been found for configurations with different fixed boundary spins or equivalently infinite BMF [18, 19, 20], or finite BMF [21]. These methods use mainly Jordan-Wigner transformation and spinor method related to rotation matrices [22, 23]. One interesting case is a configuration with infinite BMF on the two sides of an infinite strip with opposite sign [18]. A single wall domain pinned along the middle of the strip is present at low temperature, separating two regions of opposite magnetisation. Diffusive or sharp interfaces occur depending on temperature. Another case is when each boundary is composed of spins up followed at one point by spins down alongside of an infinite strip [24, 25]. From this point, an interface develops perpendicular to the strip inside the bulk. Other possible configurations with fixed boundary spins were studied using the same methods [20, 26]. A solution with finite BMF was also proposed using Boundary Quantum Field Theory [27], or Conformal Field Theory in continuous systems [28].

Our aim here is to present an alternative derivation of the exact partition function including a general BMF, extending the method introduced by Plechko. This method appears to be simpler than the McCoy and Wu's derivation, or transfer matrix methods, and allows exact calculations for more complicated cases of BMF with finite amplitude. By this we demonstrate that the partition function can be simply expressed as a

Grassmann path integral of a 1D Gaussian action with general or random magnetic fields after integrating over the bulk degrees of freedom. In the special case of an homogeneous BMF, we are able to compute the free energy on the lattice, and boundary spin-spin correlation functions. We check that our formula is equivalent to the one obtained by McCoy and Wu [12, 16]. Moreover, we also apply this method to obtain the exact partition function and the free energy when there is an interface developing between two finite opposite fields  $h$  and  $-h$  on one boundary side, as the simplest application of an inhomogeneous BMF. To our knowledge, this case has not been considered in the literature, except for infinite fields [18], and is very similar to a problem of an interface pinning on a boundary with the strength of the pinning that can be tuned (here the magnitude of the field). This method can be easily extended for more general configurations and therefore is useful for instance to studying wetting problems [18, 29]. The article is organised as follow: In section 2, we introduce the notations used throughout the article. In section 3 and 4, following Plechko's method, we obtain a Grassmann path integral representation of the partition function and the main action. In section 5, as a preliminary and useful exercise, we solve in the same way the 1D Ising model with an homogeneous magnetic field, in order to introduce the method to the reader. Section 6 is dedicated to the explicit calculation of the partition function for the two dimensional Ising model; we then give the corresponding boundary 1D action after integrating over the bulk variables. At this stage, we compare our results with those of McCoy and Wu, and Au-Yang and Fisher, taking the thermodynamic limit. In the section 7, we compute the two-point correlation function on the boundary, which is necessary, in section 8, to obtain the expressions of the partition function and the free energy in the case of a pinned interface on one boundary with an inhomogeneous BMF.

## 2. General notation

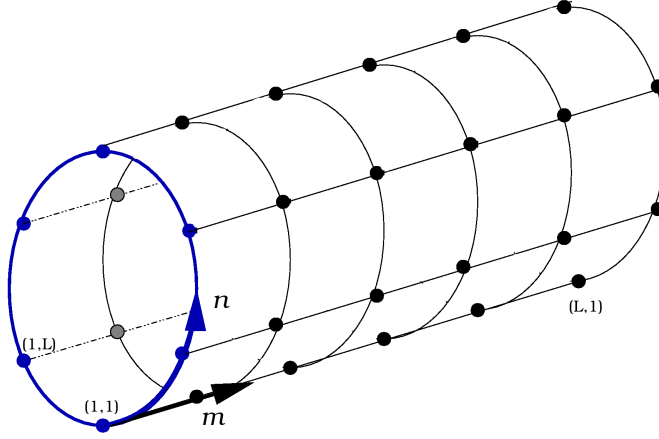
In the following we consider the Ising model on a square lattice of size  $L$  with spins  $\sigma_{mn} = \pm 1$ . For sake of simplicity, we limit ourself here to the case where the coupling constant  $J$  is the same in both directions. The method works however if there are two different coupling constants along vertical and horizontal bounds. Until section 4 we consider inhomogeneous magnetic fields  $h_n$ , placed on the sites of the first column  $m = 1$  (see Figure 1). Periodic boundary conditions for the spins are imposed in the direction parallel to the magnetic field line,  $\sigma_{m1} = \sigma_{mL+1}$ , and free boundary conditions in the transverse direction, formally equivalent to  $\sigma_{0n} = \sigma_{L+1n} = 0$ .

The Hamiltonian is then given by

$$\mathcal{H} = -J \sum_{m,n=1}^L (\sigma_{mn}\sigma_{m+1n} + \sigma_{mn}\sigma_{mn+1}) - \sum_{n=1}^L h_n \sigma_{1n}. \quad (1)$$

The partition function (PF)  $\mathcal{Z}$  is defined as

$$\mathcal{Z} = \sum_{\{\sigma_{mn}\}} \exp(-\beta \mathcal{H}),$$



**Figure 1.** Description of the model on a periodical lattice with free conditions in one direction.

where  $\beta = 1/k_B T$  and the sum is over all possible spin configurations. We can write, using  $\sigma_{mn}^2 = 1$ ,

$$e^{-\beta \mathcal{H}} = [\cosh(\beta J)]^{2L^2} \prod_{n=1}^L \cosh(\beta h_n) (1 + u \sigma_{1n}) \cdot \prod_{m,n=1}^L (1 + t \sigma_{mn} \sigma_{m+1n}) (1 + t \sigma_{mn} \sigma_{mn+1}),$$

with  $u_n = \tanh(\beta h_n)$  and  $t = \tanh(\beta J)$ . We then define  $\mathcal{Q}$  as

$$\mathcal{Q}[h] = \text{Tr}_{\sigma_{mn}} \left[ \prod_{m,n=1}^L (1 + t \sigma_{mn} \sigma_{m+1n}) (1 + t \sigma_{mn} \sigma_{mn+1}) \cdot \prod_{n=1}^L (1 + u_n \sigma_{mn}) \right], \quad (2)$$

where  $\text{Tr}_{\sigma}$  is the normalised sum  $\frac{1}{2} \sum_{\sigma=\pm 1}$ . This PF has already been calculated by McCoy and Wu [12] for a uniform field on a boundary. This was done in two steps. First they proved that this PF is the Pfaffian of a matrix, using dimer statistics, they then performed the direct calculation of this Pfaffian. Here we treat the problem in a different way. Our idea is to generalise the elegant method introduced by Plechko [8, 9] for the 2D Ising model in zero field since this method appears to be simple, and provides a direct link with Quantum Field Theory. It is indeed straightforward to obtain the expression of the quadratic fermionic action. Here we show that we can derive such a quadratic action in presence of a general boundary magnetic field.

### 3. Transformation of the PF using a Grassmann representation

Following Plechko, we introduce pairs of Grassmann variables in order to remove the local interaction between spins. We briefly define some useful tools using Grassmann algebra. For more details, we refer the reader to the book by Nakahara [30].

A Grassmann algebra  $\mathcal{A}$  of size  $N$  is a set of  $N$  anti-commuting objects  $\{a_i\}_{i=1,N}$  satisfying:

$$\forall 1 \leq i, j \leq N, \quad a_i a_j = -a_j a_i,$$

which implies  $a_i^2 = 0$ . Functions defined on such an algebra are particularly simple, they are polynomials. It is possible to define a notion of integration [30] with the following rules:

$$\int da \, a = 1, \int da \, 1 = 0, \quad (3)$$

and for any function  $f(a)$ ,

$$\int da \, f(a) = \frac{\partial f(a)}{\partial a}. \quad (4)$$

With these definitions, Gaussian integrals are expressed by

$$\int \prod_{i=1}^N da_i^* da_i \exp \left( \sum_{i,j=1}^N a_i M_{ij} a_j^* \right) = \det M. \quad (5)$$

We also define a trace operator over an algebra  $\mathcal{A} = \{a, a^*\}$  as

$$\mathfrak{Tr}_{a,a^*} [f(a, a^*)] \equiv \int da^* da \, f(a, a^*) e^{aa^*}, \quad (6)$$

with the simple rules

$$\mathfrak{Tr}_{a,a^*} [1] = 1, \quad \mathfrak{Tr}_{a,a^*} [aa^*] = 1.$$

This operator will be useful in the following, and the subscripts may be omitted implicitly when the trace is performed over the Grassmann variables that are present in the expressions inside the brackets. Grassmann variables are introduced in the PF in order to decouple the spin variables. Terms containing the same spin are then put together and the sum over the spin configurations is performed. We use the fact that

$$1 + t\sigma\sigma' = \int da^* da (1 + a\sigma)(1 + ta^*\sigma') e^{aa^*} = \mathfrak{Tr}[(1 + a\sigma)(1 + ta^*\sigma')] \quad (7)$$

and follow closely the notation in reference [8]. We consider the following link variables

$$\begin{aligned} \psi_{mn}^{(1)} &= 1 + t\sigma_{mn}\sigma_{m+1n}, \\ \psi_{mn}^{(2)} &= 1 + t\sigma_{mn}\sigma_{mn+1}. \end{aligned} \quad (8)$$

In order to decouple the products of two spins, we can express each object as a trace operator over a product of two Grassmann polynomials using equation (7)

$$\begin{aligned} \psi_{mn}^{(1)} &= \mathfrak{Tr} [A_{mn} A_{m+1n}^*], \\ \psi_{mn}^{(2)} &= \mathfrak{Tr} [B_{mn} B_{mn+1}^*], \end{aligned} \quad (9)$$

where

$$\begin{aligned} A_{mn} &= 1 + a_{mn}\sigma_{mn}, \quad A_{mn}^* = 1 + ta_{m-1n}^*\sigma_{mn}, \\ B_{mn} &= 1 + b_{mn}\sigma_{mn}, \quad B_{mn}^* = 1 + tb_{mn-1}^*\sigma_{mn}. \end{aligned} \quad (10)$$

The next step is to gather the different terms corresponding to the same spin. For that we consider the mirror symmetries introduced by Plechko [9] and the operations of moving Grassmannian objects. Such operations are possible only within the trace operator. For example, the functions  $\psi_{mn}^{(1)}$  and  $\psi_{mn}^{(2)}$  are real functions, and therefore

commute with each other. We can also write  $\psi_{mn}^{(1)}\psi_{mn}^{(2)} = \mathfrak{Tr} [(A_{mn}A_{m+1n}^*)(B_{mn}B_{mn+1}^*)]$ . Inside the brackets  $[\dots]$ , the two groups  $(A_{mn}A_{m+1n}^*)$  and  $(B_{mn}B_{mn+1}^*)$  can be moved independently since we can perform the trace on each of them separately, which gives real and therefore commuting quantities like  $\psi_{mn}^{(1)}$  or  $\psi_{mn}^{(2)}$  that can be moved and inserted everywhere, and, at the end of these operations, we reintroduce the different integrations or trace function. For example we have

$$\begin{aligned}\psi_{mn}^{(1)}\psi_{mn}^{(2)} &= \psi_{mn}^{(2)}\psi_{mn}^{(1)}, \\ &= \mathfrak{Tr} [(A_{mn}A_{m+1n}^*)(B_{mn}B_{mn+1}^*)] = \mathfrak{Tr} [(B_{mn}B_{mn+1}^*)(A_{mn}A_{m+1n}^*)], \\ &= \mathfrak{Tr} [(B_{mn}(A_{mn}A_{m+1n}^*)B_{mn+1}^*)] = \mathfrak{Tr} [(A_{mn}(B_{mn}B_{mn+1}^*)A_{m+1n}^*)],\end{aligned}\tag{11}$$

where the brackets  $(\dots)$  define the commuting group of Grassmannian objects. In general, if we have three groups of commuting objects  $\{(\mathcal{O}_i\mathcal{O}_i^*)\}_{i=1..3}$  we easily obtain the following mirror symmetries that can be applied to the objects (10):

$$\begin{aligned}(\mathcal{O}_1^*\mathcal{O}_1)(\mathcal{O}_2^*\mathcal{O}_2)(\mathcal{O}_3^*\mathcal{O}_3) &= (\mathcal{O}_1^*(\mathcal{O}_2^*(\mathcal{O}_3^*\mathcal{O}_3)\mathcal{O}_2)\mathcal{O}_1) \\ &= (\mathcal{O}_3^*(\mathcal{O}_2^*(\mathcal{O}_1^*\mathcal{O}_1)\mathcal{O}_2)\mathcal{O}_3).\end{aligned}\tag{12}$$

It is also important to treat the spin boundary conditions separately from the bulk quantities in order to obtain an expression valid not only in the thermodynamic limit, but for all finite values of  $N$ . Indeed, in the direction parallel to the line of magnetic fields where  $\sigma_{mL+1} = \sigma_{m1}$ , the corresponding link element can be expressed as

$$\psi_{mL}^{(2)} = \mathfrak{Tr} [B_{mL}B_{mL+1}^*] = \mathfrak{Tr} [B_{m1}^*B_{mL}],\tag{13}$$

where  $B_{m1}^* = 1 + tb_{m0}^*\sigma_{m1}$ . The equality (13) associated with definitions (10) imposes  $b_{m0}^* = -b_{mL}^*$ . The periodic conditions on spins therefore lead to anti-periodic conditions on Grassmann variables. In the transverse direction, we have  $\sigma_{0n} = \sigma_{L+1n} = 0$ , corresponding to free boundary conditions. This implies the boundary conditions on Grassmann variables  $a_{0n}^* = 0$  and therefore  $A_{1n}^* = 1$ . The PF  $\mathcal{Q}[h]$  (2) can be written in terms of the  $\psi_{mn}^{(k)}$  as

$$\mathcal{Q}[h] = \text{Tr}_{\sigma_{mn}} \left[ \prod_{m,n=1}^L \psi_{mn}^{(1)}\psi_{mn}^{(2)} \cdot \prod_{n=1}^L (1 + u_n\sigma_{mn}) \right].\tag{14}$$

Using the mirror symmetry (12), the boundary terms can be written as

$$\begin{aligned}\prod_{n=1}^L \psi_{Ln}^{(1)} &= \mathfrak{Tr} \left[ \overrightarrow{\prod_{n=1}^L A_{Ln}} \right], \\ \prod_{m=1}^L \psi_{mL}^{(2)} &= \mathfrak{Tr} \left[ \overrightarrow{\prod_{m=1}^L B_{m1}^*} \cdot \overleftarrow{\prod_{m=1}^L B_{mL}} \right],\end{aligned}\tag{15}$$

where we introduce the notation

$$\overrightarrow{\prod_{k=1}^N A_k} = A_1 \cdot A_2 \cdots A_N,$$

$$\overleftarrow{\prod}_{k=1}^N A_k = A_N \cdot A_{N-1} \cdots A_1.$$

These products can be reorganised as follow [9]:

$$\prod_{n=1}^L \psi_{Ln}^{(1)} \cdot \prod_{m=1}^L \psi_{mL}^{(2)} = \mathfrak{Tr} \left[ \overrightarrow{\prod}_{m=1}^L B_{m1}^* \cdot \overrightarrow{\prod}_{n=1}^L A_{Ln} \cdot \overleftarrow{\prod}_{m=1}^L B_{mL} \right]. \quad (16)$$

For the bulk elements, we obtain the following arrangement

$$\prod_{m=1}^L \psi_{mn}^{(2)} = \mathfrak{Tr} \left[ \overleftarrow{\prod}_{m=1}^L B_{mn} \cdot \overrightarrow{\prod}_{m=1}^L B_{mn+1}^* \right], \quad (17)$$

$$\prod_{n=1}^{L-1} \prod_{m=1}^L \psi_{mn}^{(2)} \cdot \prod_{m=1}^L \psi_{mL}^{(2)} = \mathfrak{Tr} \left[ \overrightarrow{\prod}_{m=1}^L B_{m1}^* \cdot \prod_{n=1}^{L-1} \prod_{m=1}^L \psi_{mn}^{(2)} \cdot \overleftarrow{\prod}_{m=1}^L B_{mL} \right], \quad (18)$$

$$\prod_{n=1}^L \prod_{m=1}^L \psi_{mn}^{(2)} \cdot \prod_{n=1}^L \psi_{Ln}^{(1)} = \mathfrak{Tr} \left[ \overrightarrow{\prod}_{n=1}^L \left( \overrightarrow{\prod}_{m=1}^L B_{mn}^* \cdot A_{Ln} \cdot \overleftarrow{\prod}_{m=1}^L B_{mn} \right) \right], \quad (19)$$

where we use the fact that  $\psi_{Ln}^{(1)}$  are commuting objects as well as the product reorganisation:

$$\mathcal{O}_1^* (\mathcal{O}_1 \mathcal{O}_2^*) (\mathcal{O}_2 \mathcal{O}_3^*) \mathcal{O}_3 = \overrightarrow{\prod}_{m=1}^3 \mathcal{O}_m^* \mathcal{O}_m. \quad (20)$$

We now insert the product over the remaining  $\psi_{mn}^{(1)}$  inside the previous expression (19)

$$\mathfrak{Tr} \left[ \prod_{m=1}^{L-1} \psi_{mn}^{(1)} \cdot \overrightarrow{\prod}_{m=1}^L B_{mn}^* \cdot A_{Ln} \cdots \right] = \mathfrak{Tr} \left[ B_{1n}^* A_{1n} \cdot \overrightarrow{\prod}_{m=2}^L A_{mn}^* B_{mn}^* A_{mn} \cdots \right], \quad (21)$$

and finally obtain

$$\prod_{m,n=1}^L \psi_{mn}^{(1)} \psi_{mn}^{(2)} = \mathfrak{Tr} \left[ \overrightarrow{\prod}_{n=1}^L B_{1n}^* A_{1n} \left( \overrightarrow{\prod}_{m=2}^L A_{mn}^* B_{mn}^* A_{mn} \cdot \overleftarrow{\prod}_{m=2}^L B_{mn} \right) B_{1n} \right]. \quad (22)$$

The PF is rewritten in a way that the sum on each spin can be performed by iteration

$$\mathcal{Q}[h] = \text{Tr}_{\sigma_{mn}} \mathfrak{Tr} \left[ \overrightarrow{\prod}_{n=1}^L B_{1n}^* A_{1n} (1 + u_n \sigma_{1n}) \left( \overrightarrow{\prod}_{m=2}^L A_{mn}^* B_{mn}^* A_{mn} \cdot \overleftarrow{\prod}_{m=2}^L B_{mn} \right) B_{1n} \right]. \quad (23)$$

In fact, as we do not yet fermionize  $(1 + u_n \sigma_{1n})$ , we reproduce here the Plechko's derivation in the special case of free-periodic boundary conditions [8]. This expression is the basis of the rest of this paper. The sum over the spins outside the magnetic field region will lead to a quadratic action over Grassmann variables which therefore commute with the rest of the elements belonging to the first column  $(1, n)$ .



#### 4. Grassmannian representation of the action on the lattice

The key point in equation (23) is that the trace over the spin configurations is performed in an iterative way. For example, the first summation is done on spins  $\sigma_{Ln}$  in the products  $A_{Ln}^* B_{Ln}^* A_{Ln} B_{Ln}$ . This operation leads to a quantity which is quadratic in Grassmann variables, which can be put outside the general product (23). The same operation is then performed on spins  $\sigma_{L-1n}$  and so on. This makes Plechko's method efficient for the 2D Ising model in zero field. With a uniform magnetic field in the bulk, the spin trace over the product of the four previous operators would lead to a quantity which is linear and quadratic in Grassmann variables and does not commute with the other products. However a BMF affects only the last products depending on spins  $\sigma_{1n}$  and this makes the problem very *similar to a 1D Ising model in a uniform magnetic field*: It is therefore solvable.

##### 4.1. Trace over spins inside the bulk

For spins  $\sigma_{mn}$  inside the bulk,  $1 \leq n \leq L$  and  $2 \leq m \leq L$ , we have to evaluate successively

$$\text{Tr}_{\sigma_{mn}} [A_{mn}^* B_{mn}^* A_{mn} B_{mn}] = \exp(Q_{mn}),$$

with

$$Q_{mn} = a_{mn} b_{mn} + t^2 a_{m-1n}^* b_{mn-1}^* + t(a_{m-1n}^* + b_{mn-1}^*)(a_{mn} + b_{mn}). \quad (24)$$

These terms commute with all Grassmannian terms and can be pulled out of the remaining products. We obtain

$$\text{Tr}_{\{\sigma_{mn}\}_{m=2..L}} \left[ \prod_{m=2}^{\overrightarrow{L}} A_{mn}^* B_{mn}^* A_{mn} \cdot \prod_{m=2}^{\overleftarrow{L}} B_{mn} \right] = \exp \left( \sum_{m=2}^L Q_{mn} \right). \quad (25)$$

The PF can now be written as

$$\mathcal{Q}[h] = \mathfrak{Tt} \left[ \exp \left( \sum_{n=1}^L \sum_{m=2}^L Q_{mn} \right) \cdot \prod_{n=1}^{\overrightarrow{L}} \underbrace{\text{Tr}_{\sigma_{1n}} ((1 + u_n \sigma_{1n}) B_{1n}^* A_{1n} B_{1n})}_{\text{boundary spins } \sigma_{1n}} \right]. \quad (26)$$

##### 4.2. Trace over the boundary spins

In the expression (26), we can evaluate separately the trace over the spins  $\sigma_{1n}$ , leading to

$$\begin{aligned} \text{Tr}_{\sigma_{1n}} [(1 + u_n \sigma_{1n}) B_{1n}^* A_{1n} B_{1n}] &= 1 + a_{1n} b_{1n} + t b_{1n-1}^* (a_{1n} + b_{1n}) + u_n \mathbb{L}_n + u_n t b_{1n-1}^* a_{1n} b_{1n}, \\ \mathbb{L}_n &= a_{1n} + b_{1n} + t b_{1n-1}^*. \end{aligned} \quad (27)$$

The presence of a magnetic field on a site introduces a linear Grassmann term. This term no longer commutes with the others, and we need to compute the product in (26)

carefully.

We would like to change artificially the fixed boundary conditions to periodic ones in order to simplify the subsequent calculations based on Fourier transformation. The quadratic part of equation (27) is equal to  $\mathbb{Q}_{1n}$  with fixed boundary conditions: We can write

$$\begin{aligned}\mathbb{Q}_{1n} &= \mathbb{Q}_{1n}^p - ta_{Ln}^* \mathbb{L}_n, \\ \mathbb{Q}_{1n}^p &= a_{1n} b_{1n} + t^2 a_{0n}^* b_{1n-1}^* + t(a_{0n}^* + b_{1n-1}^*)(a_{1n} + b_{1n}),\end{aligned}\tag{28}$$

where we introduce the boundary quantities  $a_{0n}^* = a_{Ln}^*$ .  $\mathbb{Q}_{1n}^p$  corresponds to periodic boundary conditions for the Grassmann variables (or anti-periodic conditions for the spins). We obtain

$$\text{Tr}_{\sigma_{1n}} [(1 + u_n \sigma_{1n}) B_{1n}^* A_{1n} B_{1n}] = \exp (\mathbb{Q}_{1n}^p + u_n \mathbb{L}_n - ta_{Ln}^* \mathbb{L}_n). \tag{29}$$

The correction to periodic conditions due to the free boundary conditions for the spins is included in  $-ta_{Ln}^* \mathbb{L}_n$ .

#### 4.3. Grassmann variables associated with the magnetic field

Here we introduce a pair of Grassmann variables  $(\tilde{h}_n, \tilde{h}_n^*)$  associated with the BMF. In the rest of the article, we will refer to it as the fermionic magnetic field. We have

$$\exp(u_n \mathbb{L}_n) = 1 + u_n \mathbb{L}_n = \int d\tilde{h}_n^* d\tilde{h}_n (1 + u_n \tilde{h}_n)(1 + \tilde{h}_n^* \mathbb{L}_n) e^{\tilde{h}_n \tilde{h}_n^*}, \tag{30}$$

therefore

$$\text{Tr}_{\sigma_{1n}} [(1 + u_n \sigma_{1n}) B_{1n}^* A_{1n} B_{1n}] = \mathfrak{Tr}_{\tilde{h}_n, \tilde{h}_n^*} \left[ \exp \left( \mathbb{Q}_{1n}^p + (\tilde{h}_n^* - ta_{Ln}^*) \mathbb{L}_n + u_n \tilde{h}_n \right) \right].$$

We now perform a translation in the fermionic magnetic field

$$H_n = \tilde{h}_n, \quad H_n^* = \tilde{h}_n^* - ta_{Ln}^*, \tag{31}$$

which leads to

$$\prod_{n=1}^{\overrightarrow{L}} \text{Tr}_{\sigma_{1n}} [(1 + u_n \sigma_{1n}) B_{1n}^* A_{1n} B_{1n}] = \mathfrak{Tr}_{H_n, H_n^*} \left[ \exp \left( \sum_{n=1}^L \mathbb{Q}_{1n}^p + H_n^* \mathbb{L}_n + H_n ta_{Ln}^* \right) \cdot \prod_{n=1}^{\overrightarrow{L}} e^{u_n H_n} \right].$$

It is also useful to write the last  $L$  products as a non local action along the boundary line

$$\prod_{n=1}^{\overrightarrow{L}} e^{u_n H_n} = \exp \left( \sum_{n=1}^L u_n H_n + \sum_{m=1}^{L-1} \sum_{n=m+1}^L u_m u_n H_m H_n \right). \tag{32}$$

#### 4.4. Fermionic action of the PF

Putting equation (32) into (26), we obtain the Grassmannian representation of the PF

$$\mathcal{Q}[h] = \int \mathcal{D}a^* \mathcal{D}a \mathcal{D}b^* \mathcal{D}b \mathcal{D}H^* \mathcal{D}H \exp \mathcal{S}[a, a^*, b, b^*, H, H^*], \tag{33}$$

with the action  $\mathcal{S}$  defined as

$$\begin{aligned} \mathcal{S} = & \sum_{mn=1}^L (\mathbb{Q}_{mn}^p + a_{mn}a_{mn}^* + b_{mn}b_{mn}^*) + \sum_{n=1}^L H_n^* \mathbb{L}_n + \sum_{n=1}^L H_n t a_{Ln}^* \\ & + \sum_{m<n} u_m u_n H_m H_n + \sum_{n=1}^L H_n H_n^*. \end{aligned} \quad (34)$$

This action can be separated into three terms

$$\mathcal{S} = \mathcal{S}_{\text{bulk}} + \mathcal{S}_{\text{int}} + \mathcal{S}_{\text{field}}, \quad (35)$$

with

$$\begin{aligned} \mathcal{S}_{\text{bulk}} &= \sum_{mn=1}^L (\mathbb{Q}_{mn}^p + a_{mn}a_{mn}^* + b_{mn}b_{mn}^*), \\ \mathcal{S}_{\text{field}} &= \sum_{m<n} u_m u_n H_m H_n + \sum_{n=1}^L H_n H_n^*, \\ \mathcal{S}_{\text{int}} &= \sum_{n=1}^L H_n^* \mathbb{L}_n + \sum_{n=1}^L H_n t a_{Ln}^*. \end{aligned} \quad (36)$$

The PF written as (33) is just a Gaussian integral over the set of variables  $(a, a^*, b, b^*, H, H^*)$ . If we first integrate over the variables  $(a, a^*, b, b^*)$  corresponding to the action  $\mathcal{S}_{\text{bulk}} + \mathcal{S}_{\text{int}}$ , we obtain a new Gaussian action depending only on fermionic magnetic field  $(H_n, H_n^*)$ . This new action is very similar to that for a one dimensional problem. Actually the way we integrate (33) is close to solving a 1D Ising model with a magnetic field and Grassmann variables. In the next section we present briefly this case since we will use similar tools later. Our method can then be checked using the transfer matrix techniques.

## 5. 1D Ising model with a homogeneous magnetic field

The treatment is similar to the 2D Ising model, except that there is only one kind of link variables and no mirror symmetry involved. The exact solution in the case of a homogeneous magnetic field  $u = \tanh(\beta h)$  using the transfer matrix method is simply

$$2^L \mathcal{Q}_{\text{1D}}(h) = \left(1 + t + \sqrt{(1-t)^2 + 4tu^2}\right)^L + \left(1 + t - \sqrt{(1-t)^2 + 4tu^2}\right)^L. \quad (37)$$

If we apply the Grassmann transformations as before, we can write an equation similar to (33):

$$\mathcal{Q}_{\text{1D}}(h) = \int \mathcal{D}a^* \mathcal{D}a \mathcal{D}H^* \mathcal{D}H \exp(\mathcal{S}_{\text{bulk}} + \mathcal{S}_{\text{int}} + \mathcal{S}_{\text{field}}), \quad (38)$$

with

$$\begin{aligned} \mathcal{S}_{\text{bulk}} &= \sum_{n=1}^L (a_n a_n^* + t a_{n-1}^* a_n), \quad \mathcal{S}_{\text{int}} = \sum_n H_n^* \mathbb{L}_n, \\ \mathcal{S}_{\text{field}} &= \sum_n H_n H_n^* + u^2 \sum_{m<n} H_m H_n, \quad \mathbb{L}_n = a_n + t a_{n-1}^*. \end{aligned}$$

The Grassmann variables  $a_n$  and  $H_n$  are anti-periodic in space, and can be Fourier transformed using  $a_n = \frac{1}{\sqrt{L}} \sum_q r_{q+\frac{1}{2}}^n a_{q+\frac{1}{2}}$  with  $r_q = e^{2i\pi q/L}$ . In the new basis, the actions are almost diagonalized

$$\begin{aligned} \mathcal{S}_{\text{bulk}} + \mathcal{S}_{\text{int}} = & \sum_{q=0}^{L/2-1} (1 - tr_{q+\frac{1}{2}}) a_{q+\frac{1}{2}} a_{q+\frac{1}{2}}^* + (1 - t\bar{r}_{q+\frac{1}{2}}) a_{-q-\frac{1}{2}} a_{-q-\frac{1}{2}}^* \\ & + \sum_{q=0}^{L/2-1} H_{q+\frac{1}{2}}^* (a_{q+\frac{1}{2}} + t\bar{r}_{q+\frac{1}{2}} a_{-q-\frac{1}{2}}^*) + \sum_{q=0}^{L/2-1} H_{-q-\frac{1}{2}}^* (a_{-q-\frac{1}{2}} + tr_{q+\frac{1}{2}} a_{q+\frac{1}{2}}^*), \end{aligned} \quad (39)$$

where the bar defines the complex conjugate. For  $L$  odd, we have to take care of the momenta on the diagonal of the Brillouin zone: it leads to additional terms in (39); however, these terms are irrelevant for large  $L$ . In the following, we restrict ourselves to  $L$  even.

The non local sum in  $\mathcal{S}_{\text{field}}$  can be written in the Fourier modes as

$$\begin{aligned} \sum_{m < n} H_m H_n &= \sum_q \frac{1}{r_{q+\frac{1}{2}} - 1} H_{q+\frac{1}{2}} H_{-q-\frac{1}{2}}, \\ &= \sum_{q=0}^{L/2-1} \left( \frac{1}{r_{q+\frac{1}{2}} - 1} - \frac{1}{\bar{r}_{q+\frac{1}{2}} - 1} \right) H_{q+\frac{1}{2}} H_{-q-\frac{1}{2}}. \end{aligned} \quad (40)$$

We can separate the previous sums (39) into independent blocks of four Grassmann variables  $(a_{q+\frac{1}{2}}, a_{q+\frac{1}{2}}^*, a_{-q-\frac{1}{2}}, a_{-q-\frac{1}{2}}^*)$ . To compute the individual block integrals, we use the general Gaussian formula:

$$\int da^* da db^* db \exp(\alpha aa^* + \bar{\alpha} bb^* + va + v^* a^* + wb + w^* b^*) = \alpha \bar{\alpha} \exp\left(-\frac{ww^*}{\bar{\alpha}} - \frac{vv^*}{\alpha}\right),$$

where  $(v, v^*, w, w^*)$  are Grassmann variables and  $(\alpha, \bar{\alpha})$  two independent complex numbers. We obtain

$$\begin{aligned} \mathcal{Q}_{1D}(h) &= \mathcal{Q}_{1D}(0) \int \mathcal{D}H^* \mathcal{D}H \exp\left(\sum_{q=0}^{L/2-1} H_{q+\frac{1}{2}} H_{q+\frac{1}{2}}^* + H_{-q-\frac{1}{2}} H_{-q-\frac{1}{2}}^* \right. \\ &\quad \left. + \gamma_{q+\frac{1}{2}}^{\text{1D}} H_{-q-\frac{1}{2}}^* H_{q+\frac{1}{2}}^* + u^2 \delta_{q+\frac{1}{2}} H_{q+\frac{1}{2}} H_{-q-\frac{1}{2}}\right), \end{aligned} \quad (41)$$

where

$$\mathcal{Q}_{1D}(0) = \prod_{q=0}^{L/2-1} |1 - tr_{q+\frac{1}{2}}|^2, \quad \gamma_{q+\frac{1}{2}}^{\text{1D}} = \frac{2it \sin \theta_{q+\frac{1}{2}}}{1 - 2t \cos \theta_{q+\frac{1}{2}} + t^2}, \quad (42)$$

$$\delta_{q+\frac{1}{2}} = -i \cot\left(\frac{1}{2}\theta_{q+\frac{1}{2}}\right), \quad \theta_{q+\frac{1}{2}} = \frac{2\pi}{L} \left(q + \frac{1}{2}\right). \quad (43)$$

$\mathcal{Q}_{1D}(0)$  is the PF in zero field, and is equal to  $1 + t^L$  in this case.

The remaining integrals over  $(H, H^*)$  are easy to evaluate, if we use

$$\int da^* da db^* db \exp(aa^* + bb^* + \alpha a^* b^* + \beta ab) = 1 - \alpha\beta. \quad (44)$$

After some algebra and simplifications, we finally obtain

$$\mathcal{Q}_{1D}(h) = \prod_{q=0}^{L/2-1} \left( 1 + t^2 - 2t \cos \theta_{q+\frac{1}{2}} + 4tu^2 \cos^2\left(\frac{1}{2}\theta_{q+\frac{1}{2}}\right) \right). \quad (45)$$

Results (37) and (45) are equivalent when  $L$  is even but are written in a different way. This has been checked numerically for finite size systems, and analytically in the thermodynamic limit.

Using Grassmann variables to express the PF in terms of products over Fourier modes is of course less efficient in the 1D case. However in 2D, the method is very similar and leads to a final expression which is similar to (45) and (41) as seen in the next section.

## 6. Explicit calculation of the PF

In this section, we perform the Gaussian integration of the Grassmannian form of  $\mathcal{Q}$  (33) : We start by reducing the number of Grassmann variables per site, then we integrate over the variables in the bulk, in order to obtain a *1D action*, expressed in terms of the fermionic magnetic field. Finally, this last integral is evaluated, leading to the PF on the lattice. In order to validate our method, we check that our result is identical that obtained by McCoy and Wu, in the thermodynamic limit.

### 6.1. Reduction of number of Grassmann variables per site

In the 2D case, we can easily integrate half of the Grassmann variables, for example  $(a_{mn}, b_{mn})$ , by using the identity

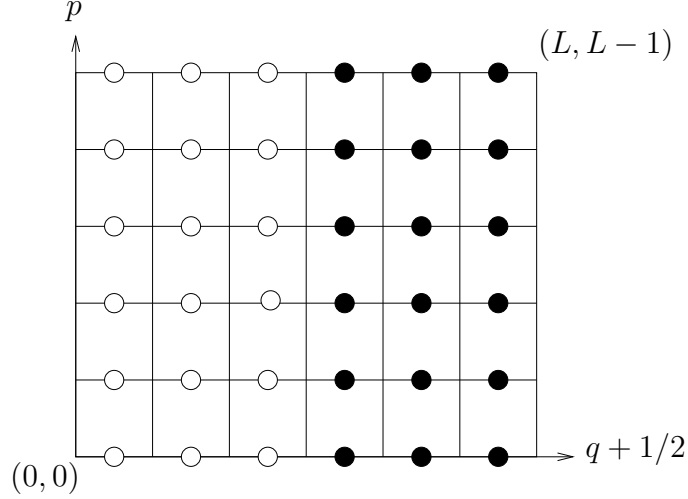
$$\int db da e^{ab+aL+b\bar{L}} = e^{\bar{L}L}. \quad (46)$$

Since the original measure is  $db^* db da^* da$ , moving  $db$  to the right of  $da^*$  implies a minus sign. After integrating over  $(a, b)$  we define  $c_{mn} = a_{mn}^*$  and  $c_{mn}^* = -b_{mn}^*$ , which removes the minus signs from the new measure  $dc^* dc$ . We thus obtain the following actions

$$\begin{aligned} \mathcal{S}_{\text{bulk}} &= \sum_{m,n=1}^L c_{mn} c_{mn}^* + t(c_{mn}^* + c_{mn})(c_{m-1n} - c_{mn-1}^*) - t^2 c_{m-1n} c_{mn-1}^*, \\ \mathcal{S}_{\text{int}} &= \sum_{n=1}^L t H_n c_{0n} + (t c_{1n-1}^* + c_{1n}^* + c_{1n}) H_n^*, \\ \mathcal{S}_{\text{field}} &= \sum_{m < n} u_m u_n H_m H_n + \sum_{n=1}^L H_n H_n^*. \end{aligned} \quad (47)$$

Taking into account the different boundary conditions, we Fourier transform these variables as in the 1D case

$$c_{mn} = \frac{1}{L} \sum_{p,q=0}^{L-1} r_p^m r_{q+\frac{1}{2}}^n c_{pq+\frac{1}{2}}, \quad c_{mn}^* = \frac{1}{L} \sum_{p,q=0}^{L-1} \bar{r}_p^m \bar{r}_{q+\frac{1}{2}}^n c_{pq+\frac{1}{2}}^*, \quad (48)$$



**Figure 2.** Integration domain for the Fourier modes in the case where  $L$  is even ( $L=6$ ). The set  $S_1$  of modes corresponds to the white points. The other modes (set  $S_2$ ) are obtained from set  $S_1$  by the symmetry  $(p, q) \rightarrow (L-p, L-q-1/2)$  modulo  $L$ .

and compute the bulk action

$$\mathcal{S}_{\text{bulk}} = \sum_{p,q=0}^{L-1} \left( 1 - t\bar{r}_p - tr_{q+\frac{1}{2}} - t^2\bar{r}_p r_{q+\frac{1}{2}} \right) c_{pq+\frac{1}{2}} c_{pq+\frac{1}{2}}^* +$$

$$-t\bar{r}_p c_{pq+\frac{1}{2}} c_{L-p, L-q-\frac{1}{2}} + tr_{q+\frac{1}{2}} c_{pq+\frac{1}{2}}^* c_{L-p, L-q-\frac{1}{2}}^*. \quad (49)$$

The result implies a coupling between modes  $(p, q + \frac{1}{2})$  and  $(L-p, L-q - \frac{1}{2}) \sim (-p, -q - \frac{1}{2})$ . As in 1D, the sum can be expressed as a sum over independent blocks containing the variables  $c_{pq+\frac{1}{2}}, c_{-p-q-\frac{1}{2}}, c_{pq+\frac{1}{2}}^*$  and  $c_{-p-q-\frac{1}{2}}^*$ . These different blocks are indeed independent if we restrict to momenta  $(p, q)$  corresponding to the white points (set  $S_1$ ) of the Brillouin zone in figure 2. In this case, the modes  $(p, q + \frac{1}{2})$  belonging to  $S_1$  and the modes  $(L-p, L-q - \frac{1}{2})$  (corresponding to  $S_2$ ) completely fill a Brillouin zone. This avoids counting the Grassmann variables twice. We also need to make the action more symmetric, which can be written as

$$\mathcal{S}_{\text{bulk}} = \sum_{p,q \in S_1} \left( \alpha_{pq+\frac{1}{2}} c_{pq+\frac{1}{2}} c_{pq+\frac{1}{2}}^* + \bar{\alpha}_{pq+\frac{1}{2}} c_{-p-q-\frac{1}{2}} c_{-p-q-\frac{1}{2}}^* + \beta_p c_{pq+\frac{1}{2}} c_{-p-q-\frac{1}{2}} \right. \quad (50)$$

$$\left. + \beta_{q+\frac{1}{2}} c_{pq+\frac{1}{2}}^* c_{-p-q-\frac{1}{2}}^* \right),$$

$$\alpha_{pq+\frac{1}{2}} = 1 - t\bar{r}_p - tr_{q+\frac{1}{2}} - t^2\bar{r}_p r_{q+\frac{1}{2}}, \quad (51)$$

$$\beta_p = t(r_p - \bar{r}_p), \quad (52)$$

and for the interacting part

$$\mathcal{S}_{\text{int}} = \frac{1}{L} \sum_{p,q \in S_1} c_{pq+\frac{1}{2}} (r_p H_{q+\frac{1}{2}}^* - t H_{-q-\frac{1}{2}}) + c_{pq+\frac{1}{2}}^* \bar{r}_p (1 + tr_{q+\frac{1}{2}}) H_{-q-\frac{1}{2}}^* +$$

$$c_{-p-q-\frac{1}{2}} (\bar{r}_p H_{-q-\frac{1}{2}}^* - t H_{q+\frac{1}{2}}) + c_{-p-q-\frac{1}{2}}^* r_p (1 + t\bar{r}_{q+\frac{1}{2}}) H_{q+\frac{1}{2}}^*. \quad (53)$$

The last action,  $\mathcal{S}_{\text{field}}$ , is the same as in 1D.

### 6.2. Integration over bulk variables : 1D action

The integration over the variables  $c$  and  $c^*$  is performed using the following identity, similar to formula (41):

$$\begin{aligned} & \int da^* da db^* db \exp(\alpha aa^* + \bar{\alpha} bb^* + \beta ab + \bar{\beta} a^* b^* + av + bw + a^* v^* + b^* w^*) \\ &= (\alpha \bar{\alpha} - \beta \bar{\beta}) \exp \left[ \frac{1}{\alpha \bar{\alpha} - \beta \bar{\beta}} (\bar{\alpha} v^* v + \alpha w^* w + \bar{\beta} v w + \beta v^* w^*) \right]. \end{aligned} \quad (54)$$

We obtain

$$\begin{aligned} \int \mathcal{D}c^* \mathcal{D}c e^{\mathcal{S}_{\text{bulk}} + \mathcal{S}_{\text{int}}} &= \prod_{p,q \in S_1} \left( \alpha_{pq+\frac{1}{2}} \bar{\alpha}_{pq+\frac{1}{2}} - \beta_p \beta_{q+\frac{1}{2}} \right) \exp \left( \gamma_{pq+\frac{1}{2}} H_{-q-\frac{1}{2}}^* H_{q+\frac{1}{2}}^* + \lambda_{pq+\frac{1}{2}} H_{q+\frac{1}{2}} H_{q+\frac{1}{2}}^* \right. \\ &\quad \left. + \bar{\lambda}_{pq+\frac{1}{2}} H_{-q-\frac{1}{2}} H_{-q-\frac{1}{2}}^* + \epsilon_{pq+\frac{1}{2}} H_{-q-\frac{1}{2}} H_{q+\frac{1}{2}} \right), \end{aligned} \quad (55)$$

where we have identified the different coefficients of the Grassmannian fields forming a quadratic action:

$$\begin{aligned} \gamma_{pq+\frac{1}{2}} &= \frac{1}{L} \frac{1}{\alpha_{pq+\frac{1}{2}} \bar{\alpha}_{pq+\frac{1}{2}} - \beta_p \beta_{q+\frac{1}{2}}} \left( -\alpha_{pq+\frac{1}{2}} (1 + t \bar{r}_{q+\frac{1}{2}}) + \bar{\alpha}_{pq+\frac{1}{2}} (1 + t r_{q+\frac{1}{2}}) \right. \\ &\quad \left. + \beta_p (1 + t r_{q+\frac{1}{2}}) (1 + t \bar{r}_{q+\frac{1}{2}}) + \bar{\beta}_{q+\frac{1}{2}} \right), \end{aligned} \quad (56)$$

and

$$\begin{aligned} \lambda_{pq+\frac{1}{2}} &= \frac{1}{L} \frac{1}{\alpha_{pq+\frac{1}{2}} \bar{\alpha}_{pq+\frac{1}{2}} - \beta_p \beta_{q+\frac{1}{2}}} \left( t r_p \beta_{q+\frac{1}{2}} + t r_p \alpha_{pq+\frac{1}{2}} (1 + t \bar{r}_{q+\frac{1}{2}}) \right), \\ \epsilon_{pq+\frac{1}{2}} &= \frac{1}{L} \frac{t^2 \beta_{q+\frac{1}{2}}}{\alpha_{pq+\frac{1}{2}} \bar{\alpha}_{pq+\frac{1}{2}} - \beta_p \beta_{q+\frac{1}{2}}}. \end{aligned} \quad (57)$$

Inside the product (55), the quantities in front of the exponentials can be simplified using cosine functions,

$$\alpha_{pq+\frac{1}{2}} \bar{\alpha}_{pq+\frac{1}{2}} - \beta_p \beta_{q+\frac{1}{2}} = (1 + t^2)^2 - 2t(1 - t^2) \left[ \cos \theta_p + \cos \theta_{q+\frac{1}{2}} \right],$$

and are invariant under the transformation  $(p, q + \frac{1}{2}) \rightarrow (L - p, L - q - \frac{1}{2})$ . In (55), the product over  $S_1$  of these coefficients define a bulk PF  $\mathcal{Q}_0$ :

$$\mathcal{Q}_0^2 = \prod_{p,q=0}^{L-1} \left[ (1 + t^2)^2 - 2t(1 - t^2) \left( \cos \theta_p + \cos \theta_{q+\frac{1}{2}} \right) \right]. \quad (58)$$

In the thermodynamic limit and in zero field, the free energy per site corresponding to  $\mathcal{Q}_0$  is equal to the one corresponding to  $\mathcal{Q}[0]$ , since the boundary conditions do not play any role on the bulk properties. In this limit, the second order phase transition occurs at a temperature given by the solutions of the equation  $(1 + t^2)^2 - 4t(1 - t^2) = 0$ , or  $t_c = \sqrt{2} - 1$ , when the cosines, in the long-wave length limit  $(p, q) \sim (0, 0)$ , approach unity. In this case the free energy is singular.

The previous coefficients  $\gamma_{pq+\frac{1}{2}}$  are not symmetrical in  $(p, q)$  since the model itself is not symmetrical in both directions. However they are antisymmetric  $\gamma_{p,q+\frac{1}{2}} = -\gamma_{L-p,L-q-\frac{1}{2}}$ .

This is useful in order to reduce the summation over the variables  $(p, q)$  in the action (55). This implies

$$\sum_{p, q \in S_1} \gamma_{pq+\frac{1}{2}} H_{-q-\frac{1}{2}}^* H_{q+\frac{1}{2}}^* = \sum_{q=0}^{L/2-1} \gamma_{q+\frac{1}{2}}^{2D} H_{-q-\frac{1}{2}}^* H_{q+\frac{1}{2}}^*,$$

with

$$\gamma_{q+\frac{1}{2}}^{2D} = \frac{1}{2} \sum_{p=0}^{L-1} \left( \gamma_{pq+\frac{1}{2}} - \gamma_{p-q-\frac{1}{2}} \right).$$

After simplification, we find that

$$\gamma_{pq+\frac{1}{2}} = \frac{1}{L} \frac{2it \sin \theta_{q+\frac{1}{2}}}{(1+t^2)^2 - 2t(1-t^2)(\cos \theta_p + \cos \theta_{q+\frac{1}{2}})}.$$

and

$$\gamma_{q+\frac{1}{2}}^{2D} = \frac{1}{L} \sum_{p=0}^{L-1} \frac{2it \sin \theta_{q+\frac{1}{2}}}{(1+t^2)^2 - 2t(1-t^2)(\cos \theta_p + \cos \theta_{q+\frac{1}{2}})}. \quad (59)$$

The factors  $\gamma^{2D}$  play the role of Fourier coefficients of an effective interaction between the boundary spins in the magnetic field. We can notice from (41) that coefficients  $\gamma^{1D}$  describe the nearest neighbour interaction of the 1D Ising model. Here the spins on the boundary can be mapped onto a 1D model.

The factors  $\lambda_{pq+\frac{1}{2}}$  have a different symmetry,  $\lambda_{pq+\frac{1}{2}} = \bar{\lambda}_{-p-q-\frac{1}{2}}$ , which allows the same kind of manipulation as before. Hence, defining

$$\begin{aligned} \Lambda_{q+\frac{1}{2}} &= \frac{1}{2} \sum_{p=0}^{L-1} (\lambda_{pq+\frac{1}{2}} + \bar{\lambda}_{p-q-\frac{1}{2}}), \\ &= \frac{t}{L} \sum_{p=0}^{L-1} \frac{(1-t^2) \cos(\theta_p) - t(1+2t \cos \theta_{q+\frac{1}{2}} + t^2)}{(1+t^2)^2 - 2t(1-t^2)(\cos \theta_p + \cos \theta_{q+\frac{1}{2}})}, \end{aligned} \quad (60)$$

we can write

$$\sum_{p, q \in S_1} \lambda_{pq+\frac{1}{2}} H_{q+\frac{1}{2}} H_{q+\frac{1}{2}}^* + \bar{\lambda}_{pq+\frac{1}{2}} H_{-q-\frac{1}{2}} H_{-q-\frac{1}{2}}^* = \sum_{q=0}^{L/2-1} \Lambda_{q+\frac{1}{2}} H_{q+\frac{1}{2}} H_{q+\frac{1}{2}}^* + \Lambda_{-q-\frac{1}{2}} H_{-q-\frac{1}{2}} H_{-q-\frac{1}{2}}^*.$$

Moreover, further reduction of the terms containing  $\epsilon_{pq+\frac{1}{2}}$  in the action (55) leads to the following simplification

$$\sum_{p, q \in S_1} \epsilon_{pq+\frac{1}{2}} H_{-q-\frac{1}{2}} H_{q+\frac{1}{2}} = \sum_{q=0}^{L/2-1} t^2 \gamma_{q+\frac{1}{2}}^{2D} H_{-q-\frac{1}{2}} H_{q+\frac{1}{2}}.$$

Finally, the problem of the boundary field is reduced to a 1D Gaussian action in Grassmann variables which is given by:

$$\begin{aligned} \mathcal{S}_{1D} &= \sum_{q=0}^{L/2-1} \left[ (1 + \Lambda_{q+\frac{1}{2}}) H_{q+\frac{1}{2}} H_{q+\frac{1}{2}}^* + (1 + \Lambda_{-q-\frac{1}{2}}) H_{-q-\frac{1}{2}} H_{-q-\frac{1}{2}}^* + \gamma_{q+\frac{1}{2}}^{2D} H_{-q-\frac{1}{2}}^* H_{q+\frac{1}{2}}^* \right. \\ &\quad \left. - t^2 \gamma_{q+\frac{1}{2}}^{2D} H_{q+\frac{1}{2}} H_{-q-\frac{1}{2}} \right] + \sum_{q, q'=0}^{L-1} \Delta_{q, q'}[u] H_{q+\frac{1}{2}} H_{q'+\frac{1}{2}}, \end{aligned} \quad (61)$$



with

$$\Delta_{q,q'}[u] = \frac{1}{L} \sum_{m < n} u_m u_n r_{q+\frac{1}{2}}^m r_{q'+\frac{1}{2}}^n.$$

This action has the same form as that for the 1D problem (39), except for the additional terms  $\Lambda$  and  $-t^2\gamma^{2D}$ , which are not present in the 1D case. This is due to the fact that  $\mathcal{Q}_0$  is not the true action in the zero field case: If we integrate (61) with respect to the Grassmann fields when  $u = 0$ , this will lead to a non zero corrective factor in front of  $\mathcal{Q}_0$ , which is however irrelevant in the thermodynamic limit (the free energy corresponding to this factor is of order  $L$  instead of  $L^2$ ).  $\mathcal{Q}_0$  is therefore not the finite size zero field partition function for the periodic/free spin boundary case. The factor comes from the free boundary conditions that restore these conditions.

Using this 1D Gaussian action, the partition function for the 2D Ising model with an inhomogeneous or random boundary magnetic fields reads :

$$\mathcal{Q}[h] = \mathcal{Q}_0 \int dH^* dH \exp(\mathcal{S}_{1D}). \quad (62)$$

In the following, we will compute the remaining Gaussian integrals in two special cases where  $\Delta_{qq'}[u]$  simplifies. The first one is the case of an homogeneous magnetic field, where  $\Delta_{qq'}[u] = u^2\delta_{q+1/2}\delta(q+q')$  with  $\delta(q-q')$  the Kronecker symbol, and  $\delta_{q+1/2}$  defined in (43). The second case corresponds to the simplest case of inhomogeneous magnetic field, when half of the boundary spins is subject to  $+H$  and the other half to  $-H$ .

### 6.3. Expression for the partition function and thermodynamic limit

In this section, we are interested in the special case of an homogeneous BMF. The previous action (61) reduces to:

$$\begin{aligned} \mathcal{S}_{1D} = & \sum_{q=0}^{L/2-1} \left[ (1 + \Lambda_{q+\frac{1}{2}}) H_{q+\frac{1}{2}} H_{q+\frac{1}{2}}^* + (1 + \Lambda_{-q-\frac{1}{2}}) H_{-q-\frac{1}{2}} H_{-q-\frac{1}{2}}^* + \right. \\ & \left. \gamma_{q+\frac{1}{2}}^{2D} H_{-q-\frac{1}{2}}^* H_{q+\frac{1}{2}}^* - t^2 \gamma_{q+\frac{1}{2}}^{2D} H_{q+\frac{1}{2}} H_{-q-\frac{1}{2}} \right] + u^2 \sum_{q=0}^{L-1} \delta_{q+\frac{1}{2}} H_{q+\frac{1}{2}} H_{-q-\frac{1}{2}}, \end{aligned}$$

with  $\delta_{q+\frac{1}{2}}$  defined in (43). The successive integrations over the blocks of Grassmann magnetic fields are easy to perform and we obtain:

$$\mathcal{Q}(h) = \mathcal{Q}_0 \prod_{q=0}^{L/2-1} Z_{q+\frac{1}{2}}(u),$$

with

$$Z_{q+\frac{1}{2}}(u) \equiv (1 + \Lambda_{q+\frac{1}{2}})(1 + \Lambda_{-q-\frac{1}{2}}) + \gamma_{q+\frac{1}{2}}^{2D} (u^2 \delta_{q+\frac{1}{2}} - t^2 \gamma_{q+\frac{1}{2}}^{2D}). \quad (63)$$

We factorize the previous expression, in order to distinguish between the boundary effect in zero field and the contribution due to the magnetic field:

$$\mathcal{Q}(h) = \mathcal{Q}_0 \prod_{q=0}^{L/2-1} \left[ (1 + \Lambda_{q+\frac{1}{2}})(1 + \Lambda_{-q-\frac{1}{2}}) - t^2(\gamma_{q+\frac{1}{2}}^{2D})^2 \right] \left[ 1 + \frac{u^2 \delta_{q+\frac{1}{2}} \gamma_{q+\frac{1}{2}}^{2D}}{(1 + \Lambda_{q+\frac{1}{2}})(1 + \Lambda_{-q-\frac{1}{2}}) - t^2(\gamma_{q+\frac{1}{2}}^{2D})^2} \right].$$

The total free energy is therefore written as

$$F(h) = -Lk_B T \ln \cosh(\beta h) - k_B T \ln \mathcal{Q}_0 + Lf_b + Lf_{\text{field}}, \quad (64)$$

with  $f_b$  a corrective free energy that is needed to restore the free boundary conditions in the direction transverse to the field,

$$\beta f_b = -\frac{1}{L} \sum_{q=0}^{L/2-1} \ln \left[ (1 + \Lambda_{q+\frac{1}{2}})(1 + \Lambda_{-q-\frac{1}{2}}) - t^2(\gamma_{q+\frac{1}{2}}^{2D})^2 \right], \quad (65)$$

and  $f_{\text{field}}$  the magnetic contribution to the free energy

$$\beta f_{\text{field}} = -\frac{1}{L} \sum_{q=0}^{L/2-1} \ln \left[ 1 + \frac{u^2 \delta_{q+\frac{1}{2}} \gamma_{q+\frac{1}{2}}^{2D}}{(1 + \Lambda_{q+\frac{1}{2}})(1 + \Lambda_{-q-\frac{1}{2}}) - t^2(\gamma_{q+\frac{1}{2}}^{2D})^2} \right]. \quad (66)$$

This decomposition is in agreement with McCoy and Wu's results[12, 16]. Indeed in the thermodynamic limit, we can use the following identity ( $a > b$ ),

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{a + b \cos \theta} = \frac{1}{\sqrt{(a-b)(a+b)}},$$

to obtain

$$\int_0^{2\pi} \frac{d\theta_p}{2\pi} \frac{1}{(1+t^2)^2 - 2t(1-t^2)(\cos \theta_p + \cos \theta_{q+\frac{1}{2}})} = \frac{1}{\sqrt{R(\theta_{q+\frac{1}{2}})}},$$

with the function  $R$  defined by

$$R(\theta) = [(1+t^2)^2 + 2t(1-t^2)(1-\cos \theta)] [(1+t^2)^2 - 2t(1-t^2)(1+\cos \theta)].$$

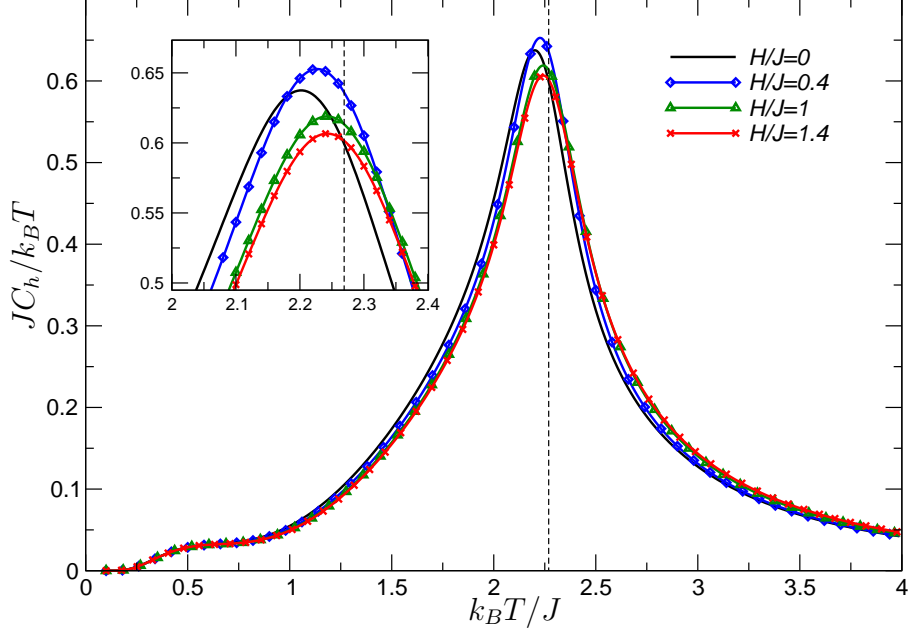
Then the following coefficients can be evaluated in this limit:

$$\gamma_{q+\frac{1}{2}}^{2D} = \frac{2it \sin \theta_{q+\frac{1}{2}}}{\sqrt{R(\theta_{q+\frac{1}{2}})}}, \quad \Lambda_{q+\frac{1}{2}} = -\frac{1}{2} + \frac{1}{2} \frac{(1+t^2)(1-2t \cos \theta_{q+\frac{1}{2}} - t^2)}{\sqrt{R(\theta_{q+\frac{1}{2}})}}. \quad (67)$$

Using the previous results and after some algebra, we recover the result of McCoy and Wu, and the continuous form for the boundary free energy depending on the magnetic field is

$$-\beta f_{\text{field}} = \frac{1}{4\pi} \int_{-\pi}^{\pi} d\theta \ln \left( 1 + \frac{4u^2 t(1+\cos \theta)}{(1+t^2)(1-2t \cos \theta - t^2) + \sqrt{R(\theta)}} \right). \quad (68)$$

The expression (66) of the free energy allows the numerical computation of the specific heat, even for  $L$  small. The results are presented on the figure 6.3 for various magnetic fields.



**Figure 3.** Specific heat for various values of homogeneous magnetic field on the boundary for  $L=20$ . The inset is a zoom in the region of the maximum. The vertical dashed line is the position of critical temperature for 2D Ising model in zero field,  $k_B T_c/J \simeq 2.26$ . See also the same field dependence in reference [21], figure 3.

## 7. Boundary two point correlation function and magnetisation

In this section, we compute the boundary spin-spin correlation functions along the boundary line between two different sites  $(1, k)$  and  $(1, l)$ ,  $k < l$ , in the case where the magnetic field is uniform. This is the easiest case since we can use simple properties of the Grassmann magnetic fields. Using (14), we have

$$\begin{aligned} \langle \sigma_{1k} \sigma_{1l} \rangle &\propto \text{Tr}_{\sigma} [\sigma_{1k} \sigma_{1l} e^{-\beta \mathcal{H}}], \\ &= \text{Tr}_{\sigma} \left[ \prod_{m,n=1}^L \psi_{mn}^{(1)} \psi_{mn}^{(2)} \cdot \prod_{n=1}^L \sigma_{1k} \sigma_{1l} (1 + u \sigma_{mn}) \right]. \end{aligned} \quad (69)$$

We then write ( $u \neq 0$ ):

$$\sigma_{1k} (1 + u \sigma_{1k}) = u \left( 1 + \frac{1}{u} \sigma_{1k} \right), \quad (70)$$

and introduce local magnetic fields  $u_n = u + (u^{-1} - u)(\delta_{kn} + \delta_{ln})$  so that

$$\langle \sigma_{1k} \sigma_{1l} \rangle \propto u^2 \text{Tr}_{\sigma} \left[ \prod_{m,n=1}^L \psi_{mn}^{(1)} \psi_{mn}^{(2)} \cdot \prod_{n=1}^L (1 + u_n \sigma_{mn}) \right].$$

We remarks that this expression is the PF of the 2D Ising model in the particular case of an inhomogeneous BMF (14). The integration over the bulk variables is not affected by this change. The difference appears *only* in the non local coupling between the fermionic magnetic fields  $H_n$ ,

$$\prod_{m=1}^{\overrightarrow{L}} e^{u_n H_n} = \exp \left[ \sum_{n=1}^L u_n H_n + \sum_{m=1}^{L-1} \sum_{n=m+1}^L u_m u_n H_m H_n \right]. \quad (71)$$

Using the expression for  $u_n$ , we obtain

$$\sum_{m < n} u_m u_n H_m H_n = u^2 \sum_{m < n} H_m H_n + (1 - u^2)(H_k L_k + H_l L_l) + \frac{1 - u^2}{u^2} H_k H_l, \quad (72)$$

$$L_k = \sum_{n=k}^L H_n - \sum_{n=1}^{k-1} H_n,$$

and therefore,

$$\prod_{n=1}^{\overrightarrow{L}} e^{u_n H_n} = \prod_{n=1}^{\overrightarrow{L}} e^{u H_n} \cdot \left[ 1 + (1 - u^2)(H_k L_k + H_l L_l) + \frac{1 - u^2}{u^2} H_k H_l + (1 - u^2)^2 H_k L_k H_l L_l \right].$$

Then the two point correlation function can be simply expressed with correlation functions  $\langle H_k H_l \rangle$ ,  $\langle H_k L_k \rangle$ , and  $\langle H_k L_k H_l L_l \rangle$

$$\begin{aligned} \langle \sigma_{1k} \sigma_{1l} \rangle &= u^2 + u^2(1 - u^2) \langle H_k L_k + H_l L_l \rangle + (1 - u^2) \langle H_k H_l \rangle \\ &\quad + u^2(1 - u^2)^2 \langle H_k L_k H_l L_l \rangle. \end{aligned}$$

The correlation functions involving a product of four Grassmann fields can be written in term of products  $\langle H_k H_l \rangle$  according to Wick's theorem:

$$\langle H_k L_k H_l L_l \rangle = \langle H_k L_k \rangle \langle H_l L_l \rangle - \langle H_k L_l \rangle \langle H_l L_k \rangle - \langle H_k H_l \rangle \langle L_k L_l \rangle. \quad (73)$$

Using a Fourier transformation, the two field correlation functions are expressed, using the usual definitions, as

$$\begin{aligned} \langle H_k H_l \rangle &= \frac{2i}{L} \sum_{q=0}^{L/2-1} \left\langle H_{q+\frac{1}{2}} H_{-q-\frac{1}{2}} \right\rangle \sin \left[ \theta_{q+\frac{1}{2}}(k - l) \right], \\ &= \frac{2}{L} \sum_{q=0}^{L/2-1} \frac{i\gamma_{q+\frac{1}{2}}^{2D}}{Z_{q+\frac{1}{2}}(u)} \sin \left[ \theta_{q+\frac{1}{2}}(k - l) \right]. \end{aligned} \quad (74)$$

Each term on the right hand side of equation (73) can then be evaluated using the previous result:

$$\begin{aligned} \langle H_k L_k \rangle \langle H_l L_l \rangle &= \left( \frac{1}{L} \sum_{q=1}^{L/2-1} \frac{i\gamma_{q+\frac{1}{2}}^{2D}}{Z_{q+\frac{1}{2}}(u)} \cot \frac{\theta_{q+\frac{1}{2}}}{2} \right)^2, \\ \langle H_k L_l \rangle &= -\frac{1}{L} \sum_{q=1}^{L/2-1} \frac{i\gamma_{q+\frac{1}{2}}^{2D}}{Z_{q+\frac{1}{2}}(u)} \frac{\cos[\theta_{q+\frac{1}{2}}(k - l + 1/2)]}{\sin(\theta_{q+\frac{1}{2}}/2)}, \\ \langle L_k L_l \rangle &= \frac{2}{L} \sum_{q=1}^{L/2-1} \frac{i\gamma_{q+\frac{1}{2}}^{2D}}{Z_{q+\frac{1}{2}}(u)} \frac{\sin[\theta_{q+\frac{1}{2}}(k - l)]}{\sin(\theta_{q+\frac{1}{2}}/2)^2}. \end{aligned} \quad (75)$$

The magnetisation can be computed the same way. Using the identity (70) we obtain

$$\langle \sigma_{1k} \rangle = u + u(1 - u^2) \langle H_k L_k \rangle. \quad (76)$$

The connected correlation function is then

$$\langle \sigma_{1k} \sigma_{1l} \rangle - \langle \sigma_{1k} \rangle \langle \sigma_{1l} \rangle = (1 - u^2) \langle H_k H_l \rangle - u^2(1 - u^2)^2 (\langle H_k L_l \rangle \langle H_l L_k \rangle + \langle H_k H_l \rangle \langle L_k L_l \rangle).$$

These correlation functions, and particularly  $\langle H_k H_l \rangle$ , can be extended for the study of more complex configurations of the boundary magnetic field. Without field ( $u = 0$ ), the correlation function between two spins  $\sigma_{1k}$  and  $\sigma_{1l}$  are simply the correlation function between the two Grassmannian fields  $H_k$  and  $H_l$  and is given by equation (74). We can extract from this the dependence of the magnetisation per spin  $m$  in the thermodynamic limit near the critical point  $T_c$ . Indeed, it is usual to define  $m^2$  as the limit of the two point correlation function for large separation  $r = |k - l|$ :

$$\lim_{r \rightarrow \infty} \langle \sigma_{1k} \sigma_{1l} \rangle = m^2. \quad (77)$$

To obtain the main contribution from (74) near  $T_c$  in the thermodynamic limit, we use the expressions (67) to compute  $Z_{q+\frac{1}{2}}(0)$  in (74) and then make an expansion around  $t_c$  in the low temperature limit ( $t > t_c$ ) of the different quantities. We first write the correlation function as an integral

$$\begin{aligned} \langle \sigma_{1k} \sigma_{1l} \rangle &= \frac{2t}{\pi} \int_0^\pi d\theta \frac{\sin(\theta) \sqrt{R(\theta)}}{S(\theta)} \sin(\theta r), \\ S(\theta) &= \frac{1}{4} \left[ \sqrt{R(\theta)} + (1 + t^2)(1 - 2t \cos \theta - t^2) \right]^2 + 4t^4 \sin^2 \theta, \end{aligned} \quad (78)$$

and expand  $R$  and  $S$  for  $\Delta t = t - t_c$  and  $\theta$  small, which is the region where the main contribution of the integral comes from. For  $R$ , we find the following expansion:

$$\begin{aligned} \sqrt{R(\theta)} &= \Delta t(1 + t_c^2)(1 + t_c + \sqrt{2}) \left[ 1 + \frac{t_c(1 - t_c^2)}{2(1 + t_c + \sqrt{2})} \tilde{\theta}^2 + \dots \right], \\ &= \Delta t(R_0 + R_2 \tilde{\theta}^2 + \dots) \end{aligned}$$

where we defined  $\tilde{\theta} \Delta t = \theta$ ,  $R_0 = (1 + t_c^2)(1 + t_c + \sqrt{2})$  and  $R_2 = t_c(1 - t_c^4)/2$ . For  $S$ , we obtain:

$$S(\theta) = \Delta t^2 \left( S_0 + S_2 \tilde{\theta}^2 + \dots \right),$$

where  $S_0$  and  $S_2$  are numerical coefficients evaluated at  $t_c$ :  $S_0 = 16(3 - 2\sqrt{2}) \simeq 2.745$ ,  $S_2 = 4t_c^4 \simeq .118$ . After some algebra, we obtain the following behaviour near  $t_c$  of the two point correlation function:

$$\langle \sigma_{1k} \sigma_{1l} \rangle \simeq \frac{2t\Delta t}{\pi} \int_0^\infty dx \frac{x R_0}{S_0(r\Delta t)^2 + S_2 x^2} \sin x. \quad (79)$$

For  $r\Delta t$  small enough, the integral is a constant  $\pi R_0/2S_2$ , and  $m$  is then proportional to  $\sqrt{\Delta t}$ , which gives the mean field exponent  $\beta = 1/2$  for the boundary magnetisation. With the presence of a small BMF, at  $T_c$ , we use equation (76) to compute directly the magnetisation:

$$\langle \sigma_{1k} \rangle \simeq u \langle H_k L_k \rangle = \frac{ut}{\pi} \int_0^\pi d\theta \frac{(1 + \cos \theta) \sqrt{R(\theta)}}{S(\theta) + 2tu^2(1 + \cos \theta) \sqrt{R(\theta)}}. \quad (80)$$

The quantities  $R$  and  $S$  are easily evaluated at  $t_c$ :

$$\begin{aligned} R(\theta) &= 16C_0(1 - \cos \theta)(3 - \cos \theta), \\ S(\theta) &= 8C_0(1 - \cos \theta) \left[ 3 - \cos \theta + \sqrt{2(1 - \cos \theta)(3 - \cos \theta)} \right], \end{aligned}$$

with  $C_0 = 17 - 12\sqrt{2}$ . When  $u$  is small, the main contribution in (80) is when the denominator is small, or when  $\theta$  is close to zero. In this case we have  $\sqrt{R(\theta)} \simeq 4\sqrt{C_0}\theta$  and  $S(\theta) \simeq 8C_0\theta^2$ . Replacing these values in (80) we obtain in the limit where  $u \ll 1$ :

$$\langle \sigma_{1k} \rangle \simeq \frac{ut}{\pi} \int_0^\pi d\theta \frac{1}{\sqrt{C_0}\theta + 2tu^2} \simeq -\frac{2t}{\pi\sqrt{C_0}} u \log u. \quad (81)$$

These results agree with McCoy and Wu's paper [17] and we can extend the method to the case of an inhomogeneous BMF, as we will see in the next section.

## 8. An example of inhomogeneous boundary magnetic field

The previous solution in the presence of a uniform BMF (configuration  $C_a$ ) on one or two sides has been presented in different publications (see for example [31], [32] and [33]), but the existence of a 1D Gaussian action allows us to solve more general configurations of the BMF. In this section, we illustrate that considering the simplest case of a non uniform field at the boundary: Between sites  $(1, 1)$  and  $(1, l)$  we impose a field  $h$ , and between sites  $(1, l+1)$  and  $(1, L)$  a field  $-h$  (configuration noted  $C_b$ ). This problem could be interesting for the study of interfaces [18, 19, 20, 34, 35, 29] and to our knowledge it has not been solved exactly for arbitrary values of  $h$  on a finite lattice.

Using our method, it appears that the evaluation of the PF  $\mathcal{Q}(h; l)$  is a simple extension of the previous calculation. Indeed, the product (73) depending on the field can be simply expressed as

$$\exp \left( \sum_{m < n} u_m u_n H_m H_n \right) = \exp \left( u^2 \sum_{m < n} H_m H_n - 2u^2 \sum_{m=1}^l H_m \sum_{n=l+1}^L H_n \right).$$

Then we have

$$\begin{aligned} \mathcal{Q}(h; l) &= \mathcal{Q}_0 \text{Tr} \left[ e^{\mathcal{S}_{1D}} \left( 1 - 2u^2 \sum_{m=1}^l \sum_{n=l+1}^L H_m H_n \right) \right], \\ &= \mathcal{Q}(h) \left( 1 - 2u^2 \sum_{m=1}^l \sum_{n=l+1}^L \langle H_m H_n \rangle_{\mathcal{S}_{1D}} \right), \end{aligned}$$

since the other terms from the exponential expansion all include the square of linear Grassmann sums and are therefore zero.

Using the Fourier transformation (74), we obtain :

$$\frac{\mathcal{Q}(h; l)}{\mathcal{Q}(h)} = 1 + \frac{4u^2}{L} \sum_{q=0}^{L/2-1} \frac{i\gamma_{q+\frac{1}{2}}^{2D}}{Z_{q+\frac{1}{2}}(u)} \frac{\sin \theta_{q+\frac{1}{2}} l}{1 - \cos \theta_{q+\frac{1}{2}}}. \quad (82)$$

The free energy difference between the 2 field configurations  $C_b$  and  $C_a$  is positive and is equal to

$$-\beta f_{\text{int}} = \ln \left( 1 + 4u^2 \frac{1}{L} \sum_{q=0}^{L/2-1} \frac{i\gamma_{q+\frac{1}{2}}^{2D}}{Z_{q+\frac{1}{2}}(u)} \frac{\sin \theta_{q+\frac{1}{2}} l}{1 - \cos \theta_{q+\frac{1}{2}}} \right). \quad (83)$$

This term is added to the total free energy (64) and for  $l = L/2$ ,  $\sin(\theta_{q+\frac{1}{2}} l)$  is simply equal to  $(-1)^q$ . If the transverse direction has an infinite size, we can use the expressions (67) to obtain

$$-\beta f_{\text{int}} = \ln \left( 1 - 8u^2 \frac{1}{L} \sum_{q=0}^{L/2-1} \frac{(-1)^q \cot(\theta_{q+\frac{1}{2}}/2)}{\sqrt{R(\theta_{q+\frac{1}{2}})} Z_{q+\frac{1}{2}}(u)} \right), \quad (84)$$

or more explicitly

$$-\beta f_{\text{int}} = \ln \left( 1 - \frac{16tu^2}{L} \sum_{q=0}^{L/2-1} \frac{(-1)^q \cot(\theta_{q+\frac{1}{2}}/2)}{(1+t^2)(1-2t \cos \theta_{q+\frac{1}{2}} - t^2) + 4tu^2(1 + \cos \theta_{q+\frac{1}{2}}) + \sqrt{R(\theta_{q+\frac{1}{2}})}} \right).$$

In the 1D Ising model, for a system of  $L$  spins with periodic boundary conditions and with a field configuration  $C_b$  identical to the 2D boundary line with  $l = L/2$ , a similar result can be obtained, either by the transfer matrix or the Grassmannian methods of the section 5. In the first case, we obtain

$$-\beta f_{\text{int}}^{1D} = \ln \left( (1-t)^2 + \frac{8tu^2[4t(1-u^2)]^{L/2}}{(1+t - \sqrt{(1-t)^2 + 4tu^2})^L + (1+t - \sqrt{(1-t)^2 + 4tu^2})^L} \right) - \ln[(1-t)^2 + 4tu^2] \quad (85)$$

and in the thermodynamic limit this leads to

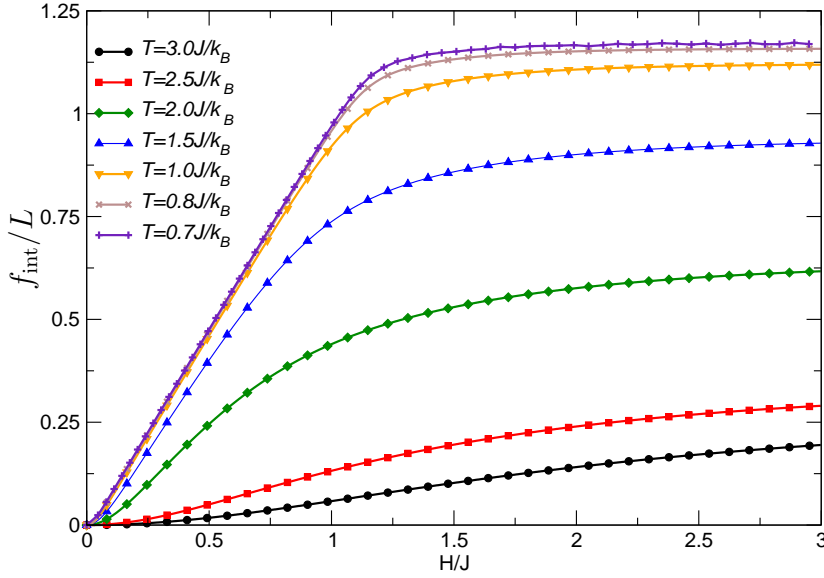
$$-\beta f_{\text{int}}^{1D} = \ln \left[ \frac{(1-t^2)^2}{(1-t)^2 + 4tu^2} \right]. \quad (86)$$

With the Grassmannian fields, we obtain a different expression but rigorously identical to equation (85):

$$-\beta f_{\text{int}}^{1D} = \ln \left( 1 - \frac{8tu^2}{L} \sum_{q=0}^{L/2-1} \frac{(-1)^q \cot(\theta_{q+\frac{1}{2}}/2)}{1+t^2 - 2t \cos(\theta_{q+\frac{1}{2}}) + 4tu^2 \cos(\theta_{q+\frac{1}{2}}/2)^2} \right). \quad (87)$$

In the thermodynamic limit and at zero temperature,  $f_{\text{int}}^{1D}$  is equal to  $4J$  for any non zero value  $h$ , which is the energy difference between the 2 ground states of  $C_b$  and  $C_a$ , all spins following the magnetic field direction in both cases. In the 2D case however, if we suppose that the size is infinite in the transverse direction, for small values of the magnetic field the boundary spins point all in the same direction (imposed by the bulk spins, up for example) below a critical value of the field  $h_c = J(1 + 4/L)$  and for the  $C_b$  configuration. Indeed the interaction between boundary spins and the ones in the bulk are strong enough that a small field  $-h$  is not sufficient to reverse these spins. In this case  $f_{\text{int}} = hL$  and is therefore extensive contrary to the 1D case. Above  $h_c$ ,

the field  $-h$  is strong enough to reverse half of the spins that were up and therefore  $f_{\text{int}} = h_c L$  due to the frustrated couplings with the neighbouring spins in the bulk. Figure 4 represents the free energy  $f_{\text{int}}/L$  as a function of the BMF for different values of the temperature.  $L = 20$  and  $h_c = 1.2J$ . There is a change of the curve slope at  $h = h_c$  and low temperature curves suggest the previous reversal picture. Figure 5 represents the contribution  $f_{\text{int}}/L$  as a function of temperature for various values of the magnetic fields. For  $h > h_c$  the curves saturate at zero temperature to the value  $f_{\text{int}}/L = h_c$  as expected. To check the low temperature behaviour of  $f_{\text{int}}$ , we can



**Figure 4.** Boundary free energy  $f_{\text{int}}/L$  as function of  $h$  for  $L = 20$ ,  $h_c = 1.2$ , for various values of the temperature. Notice the transition at  $h = h_c$  corresponding to the reversing of half the boundary spins.

perform an expansion of equation (84) for  $h \ll 1$  and  $T \ll 1$ . In this case, we assume that  $t \simeq 1$  and  $u \simeq 1 - 2 \exp(-2h/k_B T)$ . We obtain

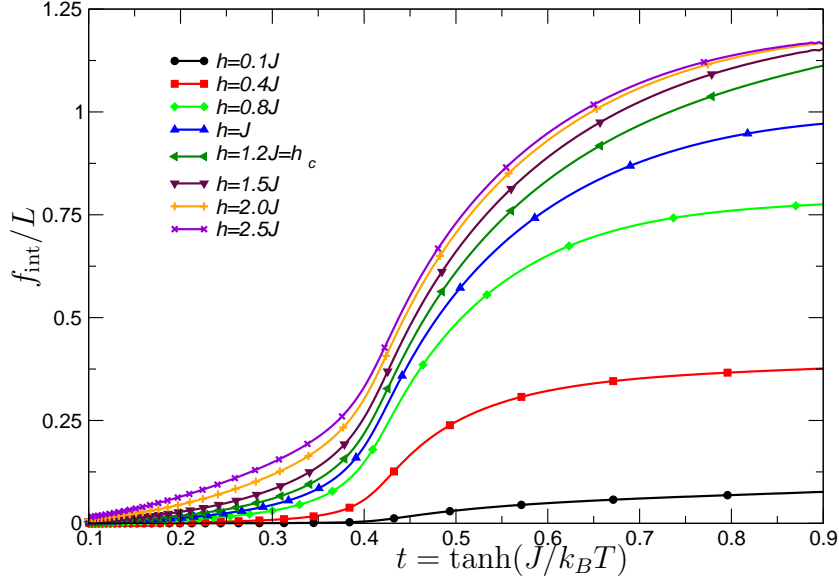
$$-\beta f_{\text{int}} \simeq \ln \left( 1 - 4u^2 \frac{1}{L} \sum_{q=0}^{L/2-1} \frac{(-1)^q \cot(\theta_{q+\frac{1}{2}}/2)}{(1+u^2) - (1-u^2) \cos \theta_{q+\frac{1}{2}}} \right). \quad (88)$$

The sum inside the logarithm can be computed using equations (85) and (87), with  $t = 1$ . We obtain

$$-\beta f_{\text{int}} = \ln \left( \frac{2(1-u^2)^{L/2}}{(1-u)^L + (1+u)^L} \right) \simeq -\beta h L \quad (89)$$

These results allow us also to study quite precisely the effect of an inhomogeneous magnetic field on the spins inside the bulk for a finite transverse size system or a fixed





**Figure 5.** Boundary free energy  $f_{\text{int}}/L$  for  $L = 20$ ,  $h_c = 1.2$ , as a function of  $\tanh(J/k_B T)$  for various values of the magnetic field  $h/J$ .

ratio between the sizes of the two directions (square or rectangular system). In this case, the previous domain wall that appears for  $h > h_c$  due to the reversal of half of the boundary spins can propagate or jump inside the bulk for sufficiently high temperature in order to lower the free energy. This may cause the total magnetisation to cancel if this wall separates two regions of equal number of opposite spins. An exhaustive study will be published in a forthcoming paper.

## 9. Conclusion

In this article we have presented a generalisation of the Plechko's method to the 2D Ising model in the case of a inhomogeneous BMF. We showed that for any configuration of the BMF, this model can be mapped onto a 1D model with a Gaussian Grassmannian action (61) similar to the simpler case of the 1D Ising in a magnetic field. Results have been obtained for the homogeneous case, in order to validate our results with the ones obtained by McCoy and Wu[16], and an extension is made to a special inhomogeneous case where an interface develops from one side of the boundary. A simple expression of the boundary partition function in term of a 1D action is given. Contrary to the 1D case, the interface free energy appears to be extensive, proportional to the system size  $L$ . A simple argument gives a zero temperature critical field  $h_c$  to be the field above which a domain wall appears on the boundary and may eventually propagate inside the bulk

at higher temperature. This generalisation of Plechko's method for the Ising model with non uniform BMF appears to be simpler than methods based on transfer matrix theories. Further developments are possible, for example in the study of wetting by a defect plane [29]. More generally, it appears however that this method does not answer the question of the 2D Ising model with an homogeneous magnetic field in the bulk [10, 11, 37, 38], even for the 3D Ising problem, since the operations using mirrors symmetries generate a Grassmannian action which is no longer quadratic. This is due to the fact, in the 2D case, that linear Grassmann terms proportional to the field do not commute each other easily. The mapping of the boundary region onto a 1D action could be used to study precisely the properties of boundary random magnetic fields. For the 2D Ising case, these random fields are found to be marginal [39] and logarithm corrections to the pure case have been checked numerically [40]. Analytical computations based on the 1D general action (61) can be performed using possibly random matrix theory (see also [41] in the context of the Conformal Field Theory).

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